APPENDIX
PROOF OF THE THEOREMS

Theorem 1. $s^i(v^x_j, v^y_j) = 1$, if and only if $s^i_{ja}(v^x_j, v^y_j) = s^i_{ja}(v^x_j, v^y_j) = 1$ for every attribute $a_j$ and when $\alpha \neq 0$ and $\alpha \neq 1$.

Proof 1. We prove its necessity first. According to Equation (9), if $s^i_{ja}(v^x_j, v^y_j) = s^i_{ja}(v^x_j, v^y_j) = 1$, then $s^i(v^x_j, v^y_j) = 1$. We prove its sufficiency by contradiction. Suppose $s^i(v^x_j, v^y_j) = 1$, then $s^i_{ja}(v^x_j, v^y_j) = s^i_{ja}(v^x_j, v^y_j) = 1$ is false. Accordingly, the true cases may be one of the following cases.

1) $s^i_{ja}(v^x_j, v^y_j) = 1$, $s^i_{ja}(v^x_j, v^y_j) \neq 1$:

$s^i(v^x_j, v^y_j) = 1$ \\
$\Rightarrow \alpha + (1 - \alpha)\frac{1}{s^i_{ja}} = 1$

Therefore, we have

$1 = \frac{1}{s^i_{ja}(v^x_j, v^y_j)} + \frac{1}{s^i_{ja}(v^x_j, v^y_j)}$.

Hence, according to Equation (9)

$\frac{1}{s^i_{ja}(v^x_j, v^y_j)} + \frac{1}{s^i_{ja}(v^x_j, v^y_j)} \geq 1 + \frac{1}{s^i_{ja}(v^x_j, v^y_j)}$.

Consequently, we conclude that the intra-attribute similarity $s^i_{ja}$ satisfies the triangle inequality.

Theorem 4. The intra-attribute similarity $s^i_{ja}$ satisfies the triangle inequality for any attribute $a_j$.

Proof 2. According to the conditions defined in Section 3, $s^i_{ja}$ satisfying the triangle inequality means that

$\frac{1}{s^i_{ja}(v^x_j, v^y_j)} + \frac{1}{s^i_{ja}(v^x_j, v^y_j)} \geq 1 + \frac{1}{s^i_{ja}(v^x_j, v^y_j)}$.

$s^i_{ja}$ satisfying the triangle inequality means that

$\frac{1}{s^i_{ja}(v^x_j, v^y_j)} + \frac{1}{s^i_{ja}(v^x_j, v^y_j)} \geq 1 + \frac{1}{s^i_{ja}(v^x_j, v^y_j)}$.

Consequently, we conclude that the intra-attribute similarity $s^i_{ja}$ satisfies the triangle inequality for any attribute $a_j$.

Theorem 5. The inter-attribute similarity $s^i_{ja}$ satisfies the triangle inequality for any attribute $a_j$.

Proof 3. We here prove that

$1 + \frac{1}{s^i_{ja}(v^x_j, v^y_j)} + \frac{1}{s^i_{ja}(v^x_j, v^y_j)} \geq 1 + \frac{1}{s^i_{ja}(v^x_j, v^y_j)}$.

Consequently, we conclude that the coupled metric attribute value similarity $s^i$ satisfies the triangle inequality.

Theorem 4. The intra-attribute similarity $s^i_{ja}$ satisfies the triangle inequality for any attribute $a_j$.

Proof 2. According to the conditions defined in Section 3, $s^i_{ja}$ satisfying the triangle inequality means that

$\frac{1}{s^i_{ja}(v^x_j, v^y_j)} + \frac{1}{s^i_{ja}(v^x_j, v^y_j)} \geq 1 + \frac{1}{s^i_{ja}(v^x_j, v^y_j)}$.

$s^i_{ja}$ satisfying the triangle inequality means that

$\frac{1}{s^i_{ja}(v^x_j, v^y_j)} + \frac{1}{s^i_{ja}(v^x_j, v^y_j)} \geq 1 + \frac{1}{s^i_{ja}(v^x_j, v^y_j)}$.

Consequently, we conclude that the intra-attribute similarity $s^i_{ja}$ satisfies the triangle inequality for any attribute $a_j$.

Theorem 5. The inter-attribute similarity $s^i_{ja}$ satisfies the triangle inequality for any attribute $a_j$.
Considering the following cases:

1) \( v^x_j = v^y_j \) or \( v^y_j = v^z_j \), or \( v^x_j = v^z_j \):

According to Equation (7) and \( s^k_{Ie} \in (0, 1] \), the following holds:

\[
\frac{1}{s^k_{Ie}(v^x_j, v^y_j)} + \frac{1}{s^k_{Ie}(v^y_j, v^z_j)} \geq 1 + \frac{1}{s^k_{Ie}(v^x_j, v^z_j)}
\]

2) \( v^x_j \neq v^y_j \) and \( v^y_j \neq v^z_j \):

\[
s^k_{Ie}(v^x_j, v^y_j) = \frac{\sum_{|W_k|} \max(p^i_x, p^i_y)}{2 \cdot \sum_{|W_k|} \max(p^i_x, p^i_y) - \sum_{|W_k|} \min(p^i_x, p^i_y)}
\]

According to the distance-similarity mapping function (see Equation (6)), the distance is:

\[
dist = 1 - \frac{\sum_{|W_k|} \min(p^i_x, p^i_y)}{\sum_{|W_k|} \max(p^i_x, p^i_y)}
\]

Note that the above is the Jaccard distance. The Jaccard distance is a metric distance and satisfies the triangle inequality. Accordingly, we conclude that \( s^k_{Ie} \) satisfies the triangle inequality.

Hence, \( s^k_{Ie} \) satisfies the triangle inequality.