APPENDIX
PROOF OF THE THEOREMS

Theorem 1. $s^j(v_j^x, v_j^y) = 1$, if and only if $s_{Ie}^j(v_j^x, v_j^y) = s_{Ia}^j(v_j^x, v_j^y) = 1$ for every attribute a_j and when $\alpha \neq 0$ and $\alpha \neq 1$.

Proof 1. We prove its necessity first. According to Equation (9), if $s_{Ie}^j(v_j^x, v_j^y) = s_{Ia}^j(v_j^x, v_j^y) = 1$, then $s^j(v_j^x, v_j^y) = 1$. We then prove its sufficiency by contradiction. Suppose $s^j(v_j^x, v_j^y) = 1$, then $s_{Ie}^j(v_j^x, v_j^y) = s_{Ia}^j(v_j^x, v_j^y) = 1$ is false. Accordingly, the true cases may be one of following cases.

1) $s_{Ie}^j(v_j^x, v_j^y) = 1, s_{Ia}^j(v_j^x, v_j^y) \neq 1$:
$s^j(v_j^x, v_j^y) = 1$
$\Leftrightarrow \alpha + (1-\alpha)\frac{1}{s_{Ia}^j} = 1$
$\Leftrightarrow s_{Ia}^j = 1 \ (\alpha \in (0,1)$
This result contradicts the assumption that $s_{Ia}^j(v_j^x, v_j^y) \neq 1$.
2) $s_{Ie}^j(v_j^x, v_j^y) \neq 1, s_{Ia}^j(v_j^x, v_j^y) = 1$:
so, $s^j(v_j^x, v_j^y) = 1$
$\Leftrightarrow \alpha\frac{1}{s_{Ie}^j} + (1-\alpha) = 1$
$\Leftrightarrow s_{Ie}^j = 1 \ (\alpha \in (0,1))$
This result contradicts the assumption that $s_{Ie}^j(v_j^x, v_j^y) \neq 1$.
3) $s_{Ie}^j(v_j^x, v_j^y) \neq 1, s_{Ia}^j(v_j^x, v_j^y) \neq 1$:
$s^j(v_j^x, v_j^y) = 1$
$\Leftrightarrow \alpha\frac{1}{s_{Ie}^j} + (1-\alpha)\frac{1}{s_{Ia}^j} = 1$
$\Leftrightarrow \alpha s_{Ia}^j + (1-\alpha)\frac{1}{s_{Ie}^j} = s_{Ie}^j s_{Ia}^j$
$\Leftrightarrow \alpha(s_{Ia}^j - s_{Ie}^j) = s_{Ie}^j s_{Ia}^j - s_{Ie}^j$
$\Leftrightarrow \alpha = \frac{s_{Ie}^j(s_{Ia}^j - 1)}{(s_{Ia}^j - s_{Ie}^j)}$
$\Leftrightarrow \frac{s_{Ie}^j(s_{Ia}^j - 1)}{(s_{Ia}^j - s_{Ie}^j)} < 1$, becuse $\alpha < 1$
Since $s_{Ie}^j \in (0,1]$ and $s_{Ia}^j \in (0,1]$
$\Leftrightarrow s_{Ie}^j(s_{Ia}^j - 1) > s_{Ia}^j - s_{Ie}^j$
$\Leftrightarrow s_{Ia}^j - 1 < \frac{s_{Ia}^j}{s_{Ie}^j} - 1$
$\Leftrightarrow s_{Ie}^j > 1$
This result contradicts that $s_{Ie}^j \leq 1$.
Hence, we conclude that $s_{Ie}^j(v_j^x, v_j^y) = s_{Ia}^j(v_j^x, v_j^y) = 1$ if $s^j(v_j^x, v_j^y) = 1$.

Theorem 3. The coupled metric attribute value similarity s^j satisfies the triangle inequality if both intra-attribute similarity s_{Ia}^j and inter-attribute similarity s_{Ie}^j satisfy the triangle inequality for every attribute a_j.

Proof 2. According to the conditions defined in Section 3, s_{Ia}^j satisfying the triangle inequality means that

$$\frac{1}{s_{Ia}^j(v_j^x, v_j^y)} + \frac{1}{s_{Ia}^j(v_j^y, v_j^z)} \geq 1 + \frac{1}{s_{Ia}^j(v_j^x, v_j^z)},$$

s_{Ie}^j satisfying the triangle inequality means that

$$\frac{1}{s_{Ie}^j(v_j^x, v_j^y)} + \frac{1}{s_{Ie}^j(v_j^y, v_j^z)} \geq 1 + \frac{1}{s_{Ie}^j(v_j^x, v_j^z)}.$$

Hence, according to Equation (9)

$$\begin{aligned}
&\frac{1}{s^j(v_j^x, v_j^y)} + \frac{1}{s^j(v_j^y, v_j^z)} \\
=&\alpha\frac{1}{s_{Ie}^j(v_j^x, v_j^y)} + (1-\alpha)\frac{1}{s_{Ia}^j(v_j^x, v_j^y)} + \\
&\alpha\frac{1}{s_{Ie}^j(v_j^y, v_j^z)} + (1-\alpha)\frac{1}{s_{Ia}^j(v_j^y, v_j^z)} \\
\geq&\alpha(1 + \frac{1}{s_{Ie}^j(v_j^x, v_j^z)}) + (1-\alpha)(1 + \frac{1}{s_{Ia}^j(v_j^x, v_j^z)}) \\
=&1 + \alpha\frac{1}{s_{Ie}^j(v_j^x, v_j^z)} + (1-\alpha)\frac{1}{s_{Ia}^j(v_j^x, v_j^z)} \\
=&1 + \frac{1}{s^j(v_j^x, v_j^z)}
\end{aligned}$$

Consequently, we conclude that the coupled metric attribute value similarity s^j satisfies the triangle inequality.

Theorem 4. The intra-attribute similarity s_{Ia}^j satisfies the triangle inequality for any attribute a_j.

Proof 3. We here prove that

$$\frac{1}{s_{Ia}^j(v_j^x, v_j^y)} + \frac{1}{s_{Ia}^j(v_j^y, v_j^z)} \geq 1 + \frac{1}{s_{Ia}^j(v_j^x, v_j^z)}$$

Considering the following cases:

1) $v_j^x = v_j^y$ or $v_j^y = v_j^z$, or $v_j^x = v_j^y = v_j^z$:
According to Equation (3) and $s_{Ia}^j \in (0,1]$, the following holds:

$$\frac{1}{s_{Ia}^j(v_j^x, v_j^y)} + \frac{1}{s_{Ia}^j(v_j^y, v_j^z)} \geq 1 + \frac{1}{s_{Ia}^j(v_j^x, v_j^z)}$$

Hence, s_{Ia}^j satisfies the triangle inequality for this case.

2) $v_j^x \neq v_j^y$ and $v_j^y \neq v_j^z$:

$$\begin{aligned}
&\frac{1}{s_{Ia}^j(v_j^x, v_j^y)} + \frac{1}{s_{Ia}^j(v_j^y, v_j^z)} - \frac{1}{s_{Ia}^j(v_j^x, v_j^z)} - 1 \\
=&\frac{\log(xy) + \log x \cdot \log y}{\log x \cdot \log y} + \frac{\log(yz) + \log y \cdot \log z}{\log y \cdot \log z} - \\
&(\frac{\log(xz) + \log x \cdot \log z}{\log x \cdot \log z} + 1) \\
=&\frac{2}{\log y}
\end{aligned}$$

Since $|I(v_j^y)| \geq 1$, $y = |I(v_j^y)| + 1 \geq 2$, accordingly $\frac{2}{\log y} \geq 0$
Therefore, we have

$$\frac{1}{s_{Ia}^j(v_j^x, v_j^y)} + \frac{1}{s_{Ia}^j(v_j^y, v_j^z)} \geq 1 + \frac{1}{s_{Ia}^j(v_j^x, v_j^z)}$$

Consequently, we conclude that the intra-attribute similarity s_{Ia}^j satisfies the triangle inequality for any attribute a_j.

Theorem 5. The inter-attribute similarity s_{Ie}^j satisfies the triangle inequality for any attribute a_j.

Proof 4. According to Equation (8), if $s_{Ie}^{k|j}$ satisfies the triangle inequality, then s_{Ie}^j satisfies it as well.

Considering the following cases:

1) $v_j^x = v_j^y$ or $v_j^y = v_j^z$, or $v_j^x = v_j^y = v_j^z$:
According to Equation (7) and $s_{Ie}^{k|j} \in (0, 1]$, the following holds:

$$\frac{1}{s_{Ie}^j(v_j^x, v_j^y)} + \frac{1}{s_{Ie}^j(v_j^y, v_j^z)} \geq 1 + \frac{1}{s_{Ie}^j(v_j^x, v_j^z)}$$

2) $v_j^x \neq v_j^y$ and $v_j^y \neq v_j^z$:

$$s_{Ie}^{k|j}(v_j^x, v_j^y) = \frac{\sum_{i=1}^{|W_k|} \max(p_x^i, p_y^i)}{2 \cdot \sum_{i=1}^{|W_k|} \max(p_x^i, p_y^i) - \sum_{i=1}^{|W_k|} \min(p_x^i, p_y^i)}$$

According to the distance-similarity mapping function (see Equation (6)), the distance is:

$$dist = 1 - \frac{\sum_{i=1}^{|W_k|} \min(p_x^i, p_y^i)}{\sum_{i=1}^{|W_k|} \max(p_x^i, p_y^i)}$$

Note that the above is the Jaccard distance. The Jaccard distance is a metric distance and satisfies the triangle inequality. Accordingly, we conclude that $s_{Ie}^{k|j}$ satisfies the triangle inequality.

Hence, s_{Ie}^j satisfies the triangle inequality.